3

The local linear trend model

The local linear trend model is obtained by adding a *slope component* ν_t to the local level model, as follows:

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \text{NID}(0, \sigma_\varepsilon^2) \\ \mu_{t+1} &= \mu_t + \nu_t + \xi_t, & \xi_t &\sim \text{NID}(0, \sigma_\xi^2) \\ \nu_{t+1} &= \nu_t + \zeta_t, & \zeta_t &\sim \text{NID}(0, \sigma_\zeta^2) \end{aligned} \tag{3.1}$$

for $t = 1, \ldots, n$. The local linear trend model contains two state equations: one for modelling the level, and one for modelling the slope. The slope ν_t in (3.1) can be conceived of as the equivalent of the regression coefficient b in classical regression model (1.1). The value of b determines the angle of the regression line with the x-axis. For the local linear trend model, the slope also determines the angle of the trend line with the x-axis. Again, the important difference is that the regression coefficient or weight b is fixed in classical regression, whereas the slope in (3.1) is allowed to change over time. In the literature on time series analysis the slope is also referred to as the *drift*.

First the results of the analysis of the UK drivers KSI with the deterministic linear trend model are presented in Section 3.1. Then in Section 3.2, the latter results are compared with those obtained with the stochastic linear trend model. Since the local linear trend model is still not the correct model for describing this time series, Section 3.4 presents the results of an analysis of the annual numbers of road traffic fatalities in Finland with the local linear trend model.

3.1. Deterministic level and slope

Fixing all state disturbances ξ_t and ζ_t in (3.1) on zero, it is easily verified that

$$\begin{aligned}
\text{for } t = 1: \qquad & y_1 = \mu_1 + \varepsilon_1, \\
& \mu_2 = \mu_1 + \nu_1 + \xi_1 = \mu_1 + \nu_1 + 0 = \mu_1 + \nu_1 \\
& \nu_2 = \nu_1 + \zeta_1 = \nu_1 + 0 = \nu_1 \\
\text{for } t = 2: \qquad & y_2 = \mu_2 + \varepsilon_2 = \mu_1 + \nu_1 + \varepsilon_2, \\
& \mu_3 = \mu_2 + \nu_2 + \xi_2 = \mu_1 + 2\nu_1 + 0 = \mu_1 + 2\nu_1 \\
& \nu_3 = \nu_2 + \zeta_2 = \nu_1 + 0 = \nu_1 \\
\text{for } t = 3: \qquad & y_3 = \mu_3 + \varepsilon_3 = \mu_1 + 2\nu_1 + \varepsilon_3, \\
& \mu_4 = \mu_3 + \nu_3 + \xi_3 = \mu_1 + 3\nu_1 + 0 = \mu_1 + 3\nu_1 \\
& \nu_4 = \nu_3 + \zeta_3 = \nu_1 + 0 = \nu_1
\end{aligned}$$

and so on.

Therefore, in this case the linear trend model simplifies to

$$y_t = \mu_1 + \nu_1 g_t + \varepsilon_t, \qquad \varepsilon_t \sim \text{NID}(0, \sigma_\varepsilon^2)$$

for $t = 1, \ldots, n$, where the predictor variable $g_t = t - 1$ for $t = 1, \ldots, n$ is effectively time, and μ_1 and ν_1 are the initial values of the level and the slope. The latter equation can also be written as

$$\begin{aligned} y_t &= (\mu_1 - \nu_1) + \nu_1(g_t + 1) + \varepsilon_t \\ &= (\mu_1 - \nu_1) + \nu_1 x_t + \varepsilon_t \end{aligned} \qquad \varepsilon_t \sim \text{NID}(0, \sigma_\varepsilon^2) \tag{3.2}$$

with $x_t = g_t + 1 = t = 1, 2, \ldots, n$.

The analysis of the log of the number of UK drivers KSI series using diffuse initialisation of the unknown values for μ_1 and ν_1 yields the following results:

```
it0    f=     0.4140728 df=1.297e-006 e1=3.742e-006 e2=4.492e-008
Strong convergence
```

Again, no iterations are required for the estimation of the parameters of this deterministic model. The value of the log-likelihood function is 0.4140728. The maximum likelihood estimate of the variance of the irregular is $\widehat{\sigma}_\varepsilon^2 = 0.022998$. The maximum likelihood estimates of the level and the slope at $t = 1$ are $\widehat{\mu}_1 = 7.5444$ and $\widehat{\nu}_1 = -0.0014480$, respectively. Substituting the latter values in (3.2) yields

$$y_t = 7.5458 - 0.00145 x_t + \varepsilon_t,$$

Table 3.1. Diagnostic tests for deterministic linear trend model and log UK drivers KSI.

	statistic	value	critical value	assumption satisfied
independence	$Q(15)$	305.680	25.00	–
	$r(1)$	0.610	±0.14	–
	$r(12)$	0.631	±0.14	–
homoscedasticity	$H(63)$	1.360	1.67	+
normality	N	1.790	5.99	+

for $t = 1, \ldots, n$ and $x_t = t$, with residual error variance $\widehat{\sigma}_\varepsilon^2 = 0.022998$, which is identical to the classical regression equation discussed in Chapter 1.

The linear trend (consisting of level plus slope) for the deterministic linear trend model is therefore identical to the regression line displayed in Figure 1.1, and the irregular for this analysis is identical to the one shown in Figure 1.3.

The results of the diagnostic tests for the residuals of the analysis are given in Table 3.1. The tests for homoscedasticity and normality are satisfactory, but the most important assumption of independence is clearly violated in this analysis.

Since one variance is estimated in model (3.2) together with two initial elements (i.e., μ_1 and ν_1), the Akaike information criterion for this model equals

$$\text{AIC} = \frac{1}{192}\left[-2(192)(0.4140728) + 2(2+1)\right] = -0.796896.$$

The deterministic linear trend model (3.2) therefore yields a better fit for the log of the number of UK drivers KSI series than the deterministic level model (see Section 2.1). However, the fit of the model is not as good as that obtained with the stochastic level model (see Section 2.2).

3.2. Stochastic level and slope

Allowing both the level and the slope to vary over time in model (3.1), the following results are obtained:

```
it0    f=     0.4839008 df=   0.04716 e1=    0.1279 e2=  0.001858
it5    f=     0.5260923 df=   0.07616 e1=    0.2568 e2=  0.005020
it10   f=     0.6215185 df=   0.01589 e1=   0.03640 e2=  0.008347
it15   f=     0.6236505 df=  0.007679 e1=   0.02624 e2=  0.002837
it20   f=     0.6247839 df=  0.002160 e1=  0.004991 e2=  0.009222
it23   f=     0.6247935 df=2.575e-006 e1=5.967e-006 e2=5.852e-006
Strong convergence
```

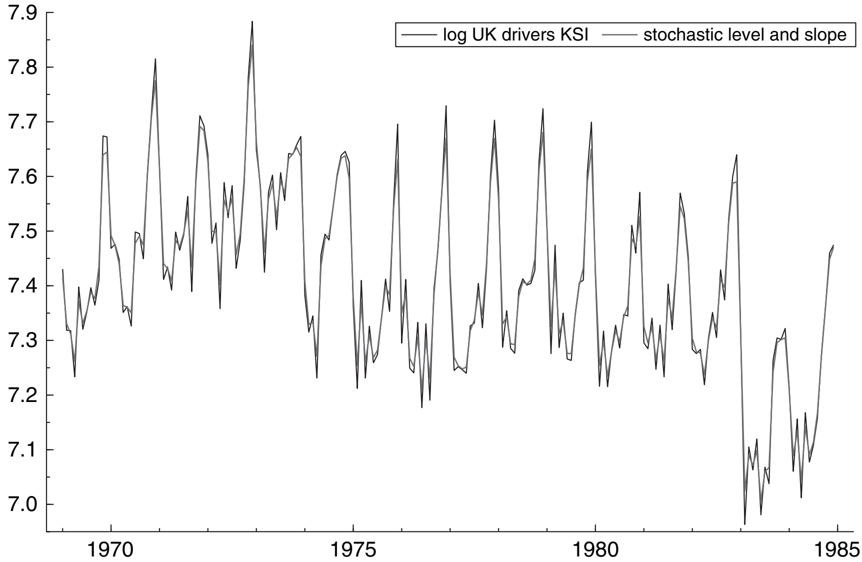

Figure 3.1. Trend of stochastic linear trend model.

At convergence the value of the log-likelihood function equals 0.6247935. The maximum likelihood estimate of the variance of the irregular is $\widehat{\sigma}^2_{\varepsilon} = 0.0021181$, and the maximum likelihood estimates of the state disturbance variances are $\widehat{\sigma}^2_{\xi} = 0.012128$ and $\widehat{\sigma}^2_{\zeta} = 1.5E^{-}11$, respectively. The maximum likelihood estimates of the initial values of the level and the slope are $\widehat{\mu}_1 = 7.4157$ and $\widehat{\nu}_1 = 0.00028896$, respectively. The state variance for the slope component is almost equal to zero, meaning that the value of the slope hardly changes over time.

The trend (consisting of level plus slope) for the stochastic linear trend model (3.1) is displayed in Figure 3.1, while Figure 3.2 contains the separate development of the slope over time. It may seem that the change of the slope over time is considerable in Figure 3.2, but when the scale on the y-axis is taken into consideration (in relation to the variation in y), it is clear that the slope is effectively constant. This is consistent with the close-to-zero disturbance variance for this component.

The irregular component for model (3.1) is displayed in Figure 3.3. The systematic pattern in the irregular of the deterministic linear trend model as observed in Figure 1.3 has largely disappeared in Figure 3.3. The values of the diagnostic tests for the residuals of the analysis are given in Table 3.2. In contrast with the previous analysis, the first autocorrelation in the correlogram ($r(1)$) is close to zero but the autocorrelation at lag 12 is

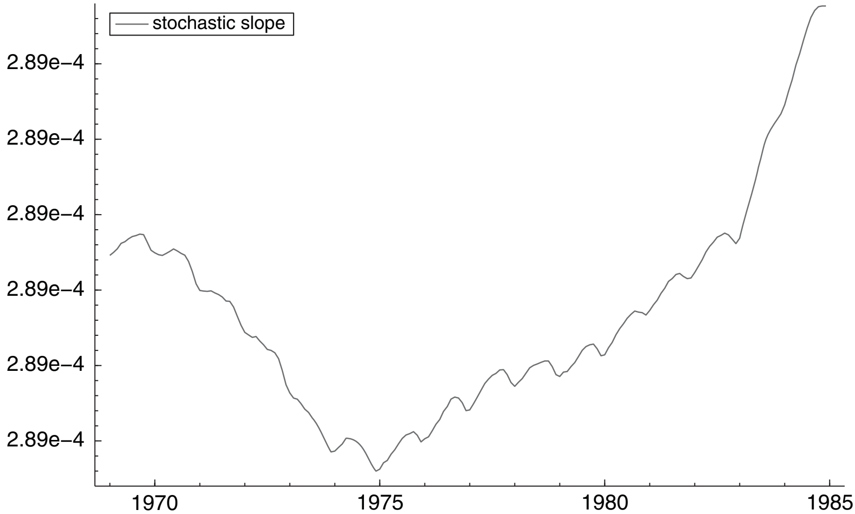

Figure 3.2. Slope of stochastic linear trend model.

still too large. The overall Q-test for the first 15 autocorrelations confirms that the assumption of independence is still not satisfied. The test for homoscedasticity is satisfactory, but here the assumption of normality is clearly violated.

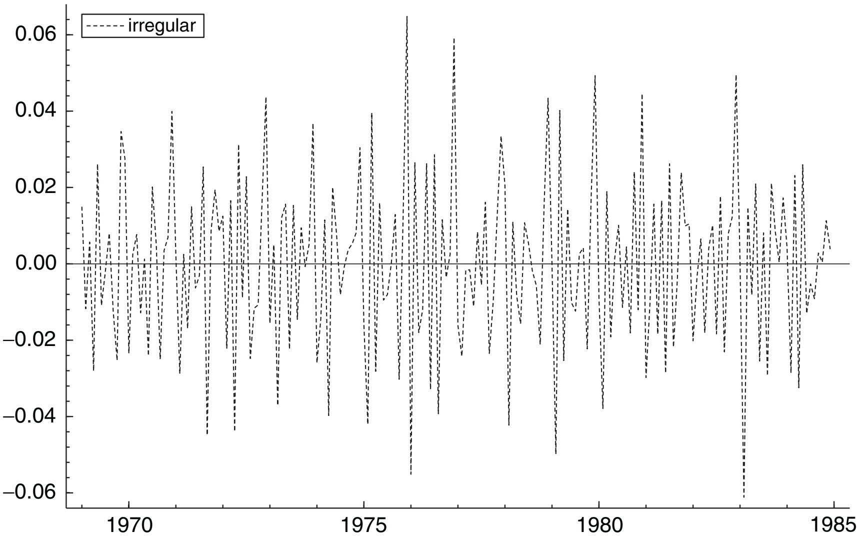

Figure 3.3. Irregular component of stochastic linear trend model.

Table 3.2. Diagnostic tests for the local linear trend model applied to the log of the UK drivers KSI.

	statistic	value	critical value	assumption satisfied
independence	$Q(15)$	100.610	22.36	–
	$r(1)$	0.005	±0.14	+
	$r(12)$	0.532	±0.14	–
homoscedasticity	$H(63)$	1.058	1.67	+
normality	N	14.946	5.99	–

The Akaike information criterion for the stochastic linear trend model equals

$$\text{AIC} = \frac{1}{192}\left[-2(192)(0.6247935) + 2(2+3)\right] = -1.1975.$$

For the log of the UK drivers KSI series the fit of the local linear trend model is inferior to that obtained with the local level model (see Section 2.2), but clearly superior to the fit obtained with a classical linear regression analysis (as modelled by the deterministic linear trend model). This suggests that the inclusion of a stochastic slope has not helped the analysis in this case.

3.3. Stochastic level and deterministic slope

Another possibility is to consider model (3.1) where only the level is allowed to vary over time whereas the slope is treated deterministically. In this case it is not very difficult to verify that model (3.1) can written as

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \text{NID}(0, \sigma_\varepsilon^2) \\ \mu_{t+1} &= \mu_t + \nu_1 + \xi_t, & \xi_t &\sim \text{NID}(0, \sigma_\xi^2) \end{aligned} \tag{3.3}$$

for $t = 1, \ldots, n$. The analysis of the log of the UK drivers KSI with model (3.3) yields the following results:

```
it0    f=    0.5432387 df=   0.08367 e1=    0.2659 e2=  0.003367
it1    f=    0.5569736 df=    0.1072 e1=    0.3318 e2=    0.3264
it2    f=    0.6210248 df=   0.05154 e1=    0.1278 e2=   0.02498
it3    f=    0.6215160 df=   0.03132 e1=   0.09584 e2=  0.002430
it4    f=    0.6224598 df=   0.02822 e1=   0.08747 e2=  0.001277
it5    f=    0.6241177 df=   0.02014 e1=   0.04977 e2=  0.003469
it6    f=    0.6246745 df=  0.007840 e1=   0.01932 e2=  0.001947
it7    f=    0.6247859 df=  0.001003 e1=  0.003322 e2=  0.001153
it8    f=    0.6247932 df= 0.0001671 e1= 0.0004376 e2= 0.0003907
it9    f=    0.6247935 df=1.173e-005 e1=2.880e-005 e2=8.883e-005
Strong convergence
```

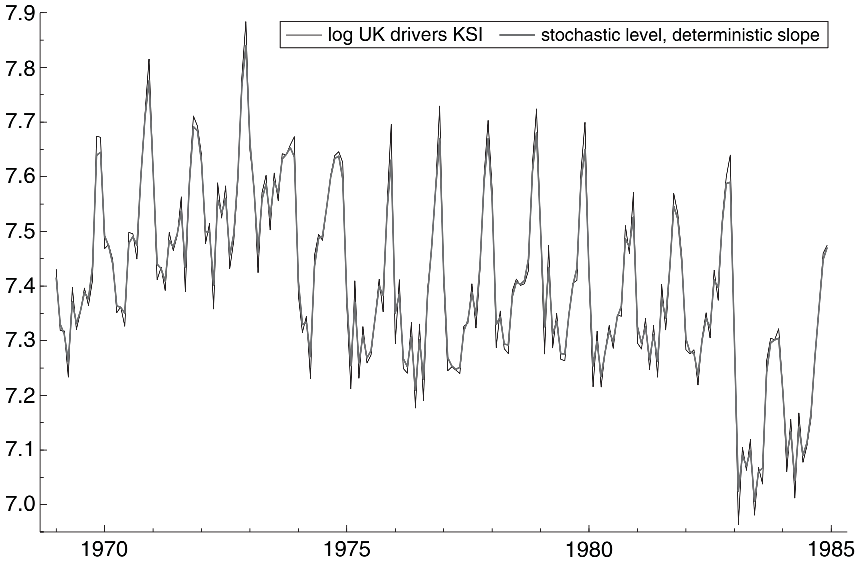

Figure 3.4. Trend of stochastic level and deterministic slope model.

At convergence the value of the log-likelihood function equals 0.6247935. The maximum likelihood estimate of the variance of the observation disturbances is $\widehat{\sigma}_{\varepsilon}^2 = 0.00211869$, and the maximum likelihood estimate of the variance of the level disturbances is $\widehat{\sigma}_{\xi}^2 = 0.0121271$. The maximum likelihood estimates of the values of the level and the slope right at the start of the series are $\widehat{\mu}_1 = 7.4157$ and $\widehat{\nu}_1 = 0.00028897$, respectively.

The trend (consisting of stochastic level and deterministic slope) is displayed in Figure 3.4. The deterministic slope is simply a constant, equal to $\widehat{\nu}_1 = 0.00028897$ for $t = 1, \ldots, n$. The irregular component for this model is virtually identical to the one in Figure 3.3, and the results of the diagnostic tests on the residuals are virtually identical to those presented in Table 3.2.

The Akaike information criterion for the linear trend model with stochastic level and deterministic slope equals

$$\text{AIC} = \frac{1}{192}\,[-2(192)(0.6247935) + 2(2 + 2)] = -1.20792.$$

Thus, the AIC of this model is slightly better than the fit of the linear trend model with stochastic level and stochastic slope. However, it is still inferior to the AIC of the stochastic level model (see Section 2.2).

It follows that the value of the variance for the slope component is almost zero and it leads to an almost negligible fluctuation in the slope (see Figure 3.2). In state space modelling, a near zero state disturbance variance indicates that the corresponding state component may as well be treated as a deterministic effect, resulting in a more parsimonious model. Treating the slope component deterministically indeed yields a slightly better fitting model. However, the fit of the latter model is still inferior to the one obtained with the local level model. This means that the addition of a slope component to the local level model is not effective in improving the description of the observed time series. Therefore, the slope is a redundant component in this case, and is removed from further analyses of the UK drivers KSI series. A similar strategy is described by Ord and Young (2004) on the basis of *t*-statistics rather than the AIC.

As the diagnostic tests in Table 3.2 indicate, the local linear trend model is still not the appropriate model for obtaining a good description of the log of the UK drivers KSI, for reasons that will be explained in Chapter 4. In the next section we therefore discuss a time series for which the local linear trend model is more appropriate.

3.4. The local linear trend model and Finnish fatalities

In this section the local linear trend model is applied to the log of the annual numbers of road traffic fatalities in Finland as observed for the years 1970 through 2003 (see Appendix B and Figure 3.5). Allowing both the level and the slope component to vary over time, at convergence the value of the log-likelihood function equals 0.7864746. The value of the AIC for this analysis therefore equals

$$\text{AIC} = \frac{1}{34}\,[-2(34)(0.7864746) + 2(2+3)] = -1.27883.$$

The maximum likelihood estimates of the variances corresponding to the irregular, level, and slope components are $\widehat{\sigma}^2_\varepsilon = 0.00320083$, $\widehat{\sigma}^2_\xi = 9.69606E^{-}26$, and $\widehat{\sigma}^2_\zeta = 0.00153314$, respectively.

Since the variance of the level disturbances is, for all practical purposes, equal to zero, the analysis is repeated with a deterministic level component, yielding the following results:

```
it0    f=     0.7544891 df=   0.07002 e1=    0.2599 e2=  0.002318
it1    f=     0.7735067 df=   0.05625 e1=    0.2050 e2=  0.003601
it2    f=     0.7858661 df=   0.01570 e1=   0.04919 e2=  0.003735
it3    f=     0.7864624 df=  0.002545 e1=  0.007951 e2= 0.0006039
```

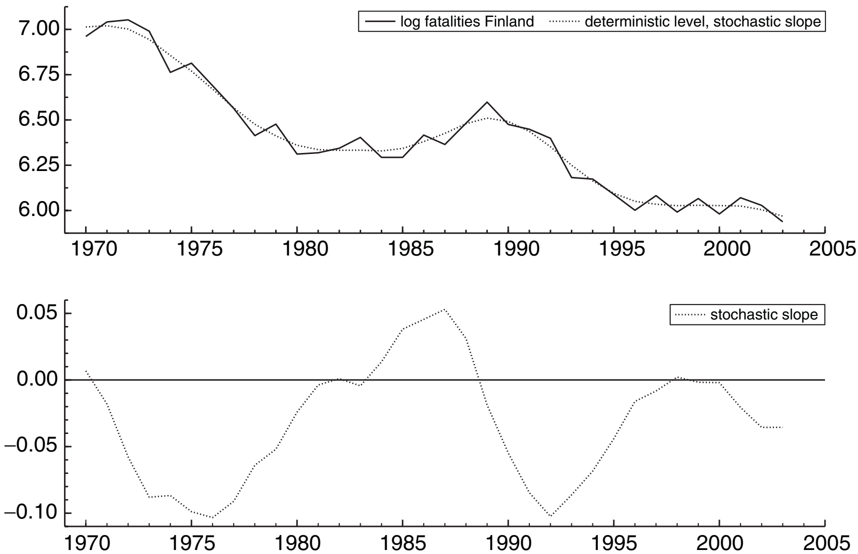

Figure 3.5. Trend of deterministic level and stochastic slope model for Finnish fatalities (top), and stochastic slope component (bottom).

```
it4    f=      0.7864746 df=4.601e-005 e1= 0.0001437 e2=6.199e-005
it5    f=      0.7864746 df=2.310e-005 e1=7.211e-005 e2=6.183e-007
Strong convergence
```

At convergence the value of the log-likelihood function equals 0.7864746. The maximum likelihood estimates of the variances of the observation and slope disturbances are $\widehat{\sigma}_{\varepsilon}^2 = 0.00320083$, and $\widehat{\sigma}_{\zeta}^2 = 0.00153314$, respectively. The maximum likelihood estimates of the values of the level and the slope at the start of the series are $\widehat{\mu}_1 = 7.0133$ and $\widehat{\nu}_1 = 0.0068482$.

The trend (consisting of a deterministic level and a stochastic slope) of this analysis is displayed at the top of Figure 3.5, while the stochastic slope is shown separately at the bottom of the figure. Since the time varying slope component in Figure 3.5 models the rate of change in the series, it can be interpreted as follows. When the slope component is *positive*, the trend in the series is *increasing*. Thus, the trend of fatalities in Finland was increasing in the years 1970, 1982, 1984 through to 1988, and in 1998 (see Figure 3.5). On the other hand, the trend is *decreasing* when the slope component is *negative*. The trend in the fatalities of Finland was therefore decreasing in the remaining years of the series.

Moreover, when the slope is positive and increasing, the increase becomes more pronounced, while the increase becomes less pronounced

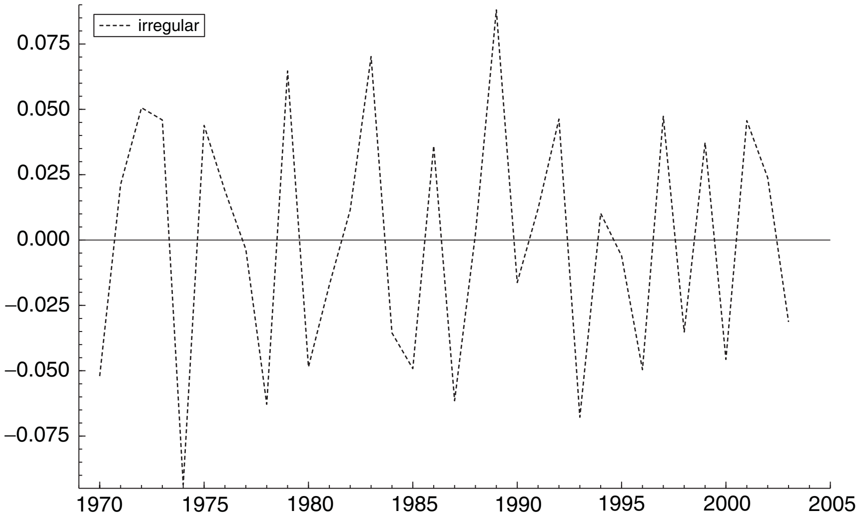

Figure 3.6. Irregular component for Finnish fatalities.

when the slope is positive but decreasing. Conversely, when the slope is negative and decreasing then the decrease becomes more pronounced, while the decrease levels off when the slope is negative but increasing.

The irregular component of the analysis is shown in Figure 3.6. The diagnostic tests for the residuals of the analysis are given in Table 3.3. Since $Q(10) < \chi^2_{(9;0.05)}$, $1/H(11) < F_{(12,12;0.025)}$, and $N < \chi^2_{(2;0.05)}$ (see also Section 8.5), the assumptions of independence, homoscedasticity, and normality are all satisfied, indicating that the deterministic level and stochastic slope model yields an appropriate description of the log of the annual traffic fatalities in Finland.

Table 3.3. Diagnostic tests for deterministic level and stochastic slope model, and log Finnish fatalities.

	statistic	value	critical value	assumption satisfied
independence	$Q(10)$	7.044	16.92	+
	$r(1)$	−0.028	±0.34	+
	$r(4)$	−0.094	±0.34	+
homoscedasticity	$1/H(11)$	1.348	3.28	+
normality	N	0.644	5.99	+

The Akaike information criterion for the deterministic level and stochastic slope model equals

$$\text{AIC} = \frac{1}{34}\,[-2(34)(0.7864746) + 2(2+2)] = -1.33766.$$

Thus, the fit of this model is slightly better than the fit of a model with stochastic level and stochastic slope. Since the log-likelihood values are identical for the two models, the improved fit of the second model can be completely attributed to its greater parsimony. The model with a deterministic level and stochastic slope is also called the *smooth trend* model, reflecting the fact that the trend of such a model is relatively smooth compared to a trend with a level disturbance variance different from zero.

As Section 3.1 illustrates, the deterministic linear trend model actually performs a classical regression analysis of time series observations on the predictor variable time. This is an important and very useful result. By way of the Akaike information criterion, it opens up the possibility of a straightforward, fair and quantitative assessment of the relative merits of state space methods and classical regression models when it comes to the analysis of time series data. The reverse is also true: the state space models discussed in the present book are regression models in which the parameters (intercept and regression coefficient(s)) are allowed to vary over time. State space models are therefore also sometimes referred to as *dynamic linear* models.