

CHAPTER 10

SEASONAL MODELS

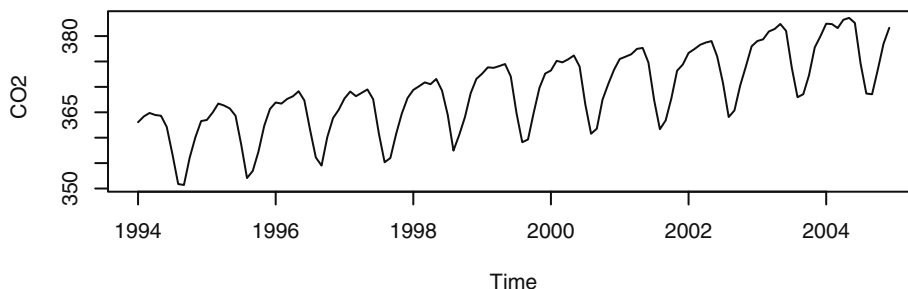
In Chapter 3, we saw how seasonal deterministic trends might be modeled. However, in many areas in which time series are used, particularly business and economics, the assumption of any deterministic trend is quite suspect even though cyclical tendencies are very common in such series.

Here is an example: Levels of carbon dioxide (CO_2) are monitored at several sites around the world to investigate atmospheric changes. One of the sites is at Alert, Northwest Territories, Canada, near the Arctic Circle.



Exhibit 10.1 displays the monthly CO_2 levels from January 1994 through December 2004. There is a strong upward trend but also a seasonality that can be seen better in the more detailed Exhibit 10.2, where only the last few years are graphed using monthly plotting symbols.

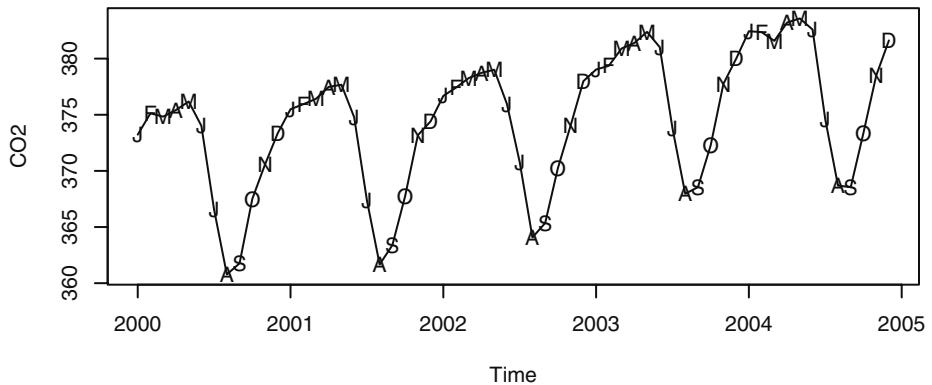
Exhibit 10.1 Monthly Carbon Dioxide Levels at Alert, NWT, Canada



```
> data(co2)
> win.graph(width=4.875,height=3,pointsize=8)
> plot(co2,ylab='CO2')
```

As we see in the displays, carbon dioxide levels are higher during the winter months and much lower in the summer. Deterministic seasonal models such as seasonal means plus linear time trend or sums of cosine curves at various frequencies plus linear time trend as we investigated in Chapter 3 could certainly be considered here. But we discover that such models do not explain the behavior of this time series. For this series and many others, it can be shown that the residuals from a seasonal means plus linear time trend model are highly autocorrelated at many lags.[†] In contrast, we will see that the **stochastic seasonal models** developed in this chapter do work well for this series.

Exhibit 10.2 Carbon Dioxide Levels with Monthly Symbols



```
> plot(window(co2,start=c(2000,1)),ylab='CO2')
> Month=c('J','F','M','A','M','J','J','A','S','O','N','D')
> points(window(co2,start=c(2000,1)),pch=Month)
```

10.1 Seasonal ARIMA Models

We begin by studying stationary models and then consider nonstationary generalizations in Section 10.3. We let s denote the known seasonal period; for monthly series $s = 12$ and for quarterly series $s = 4$.

Consider the time series generated according to

$$Y_t = e_t - \Theta e_{t-12}$$

Notice that

$$\begin{aligned} \text{Cov}(Y_p, Y_{t-1}) &= \text{Cov}(e_t - \Theta e_{t-12}, e_{t-1} - \Theta e_{t-13}) \\ &= 0 \end{aligned}$$

but that

[†] We ask you to verify this in Exercise 10.8.

$$\begin{aligned} \text{Cov}(Y_t, Y_{t-12}) &= \text{Cov}(e_t - \Theta e_{t-12}, e_{t-12} - \Theta e_{t-24}) \\ &= -\Theta \sigma_e^2 \end{aligned}$$

It is easy to see that such a series is stationary and has nonzero autocorrelations only at lag 12.

Generalizing these ideas, we define a **seasonal MA(Q) model of order Q with seasonal period s** by

$$Y_t = e_t - \Theta_1 e_{t-s} - \Theta_2 e_{t-2s} - \cdots - \Theta_Q e_{t-Qs} \quad (10.1.1)$$

with **seasonal MA characteristic polynomial**

$$\Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \cdots - \Theta_Q x^{Qs} \quad (10.1.2)$$

It is evident that such a series is always stationary and that the autocorrelation function will be nonzero only at the seasonal lags of $s, 2s, 3s, \dots, Qs$. In particular,

$$\rho_{ks} = \frac{-\Theta_k + \Theta_1 \Theta_{k+1} + \Theta_2 \Theta_{k+2} + \cdots + \Theta_{Q-k} \Theta_Q}{1 + \Theta_1^2 + \Theta_2^2 + \cdots + \Theta_Q^2} \quad \text{for } k = 1, 2, \dots, Q \quad (10.1.3)$$

(Compare this with Equation (4.2.5) on page 65 for the nonseasonal MA process.) For the model to be invertible, the roots of $\Theta(x) = 0$ must all exceed 1 in absolute value.

It is useful to note that the seasonal MA(Q) model can also be viewed as a special case of a nonseasonal MA model of order $q = Qs$ but with all θ -values zero except at the seasonal lags $s, 2s, 3s, \dots, Qs$.

Seasonal autoregressive models can also be defined. Consider

$$Y_t = \Phi Y_{t-12} + e_t \quad (10.1.4)$$

where $|\Phi| < 1$ and e_t is independent of Y_{t-1}, Y_{t-2}, \dots . It can be shown that $|\Phi| < 1$ ensures stationarity. Thus it is easy to argue that $E(Y_t) = 0$; multiplying Equation (10.1.4) by Y_{t-k} , taking expectations, and dividing by γ_0 yields

$$\rho_k = \Phi \rho_{k-12} \quad \text{for } k \geq 1 \quad (10.1.5)$$

Clearly

$$\rho_{12} = \Phi \rho_0 = \Phi \quad \text{and} \quad \rho_{24} = \Phi \rho_{12} = \Phi^2$$

More generally,

$$\rho_{12k} = \Phi^k \quad \text{for } k = 1, 2, \dots \quad (10.1.6)$$

Furthermore, setting $k = 1$ and then $k = 11$ in Equation (10.1.5) and using $\rho_k = \rho_{-k}$ gives us

$$\rho_1 = \Phi \rho_{11} \quad \text{and} \quad \rho_{11} = \Phi \rho_1$$

which implies that $\rho_1 = \rho_{11} = 0$. Similarly, one can show that $\rho_k = 0$ except at the seasonal lags 12, 24, 36, \dots . At those lags, the autocorrelation function decays exponentially like an AR(1) model.

With this example in mind, we define a **seasonal AR(P) model of order P and seasonal period s** by

$$Y_t = \Phi_1 Y_{t-s} + \Phi_2 Y_{t-2s} + \cdots + \Phi_P Y_{t-Ps} + e_t \quad (10.1.7)$$

with **seasonal characteristic polynomial**

$$\Phi(x) = 1 - \Phi_1 x^s - \Phi_2 x^{2s} - \cdots - \Phi_P x^{Ps} \quad (10.1.8)$$

As always, we require e_t to be independent of Y_{t-1}, Y_{t-2}, \dots , and, for stationarity, that the roots of $\Phi(x) = 0$ be greater than 1 in absolute value. Again, Equation (10.1.7) can be seen as a special AR(p) model of order $p = Ps$ with nonzero ϕ -coefficients only at the seasonal lags $s, 2s, 3s, \dots, Ps$.

It can be shown that the autocorrelation function is nonzero only at lags $s, 2s, 3s, \dots$, where it behaves like a combination of decaying exponentials and damped sine functions. In particular, Equations (10.1.4), (10.1.5), and (10.1.6) easily generalize to the general seasonal AR(1) model to give

$$\rho_{ks} = \Phi^k \text{ for } k = 1, 2, \dots \quad (10.1.9)$$

with zero correlation at other lags.

10.2 Multiplicative Seasonal ARMA Models

Rarely shall we need models that incorporate autocorrelation *only* at the seasonal lags. By combining the ideas of seasonal and nonseasonal ARMA models, we can develop parsimonious models that contain autocorrelation for the seasonal lags but also for low lags of neighboring series values.

Consider a model whose MA characteristic polynomial is given by

$$(1 - \theta x)(1 - \Theta x^{12})$$

Multiplying out, we have $1 - \theta x - \Theta x^{12} + \theta \Theta x^{13}$. Thus the corresponding time series satisfies

$$Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta \Theta e_{t-13} \quad (10.2.1)$$

For this model, we can check that the autocorrelation function is nonzero only at lags 1, 11, 12, and 13. We find

$$\gamma_0 = (1 + \theta^2)(1 + \Theta^2)\sigma_e^2 \quad (10.2.2)$$

$$\rho_1 = -\frac{\theta}{1 + \theta^2} \quad (10.2.3)$$

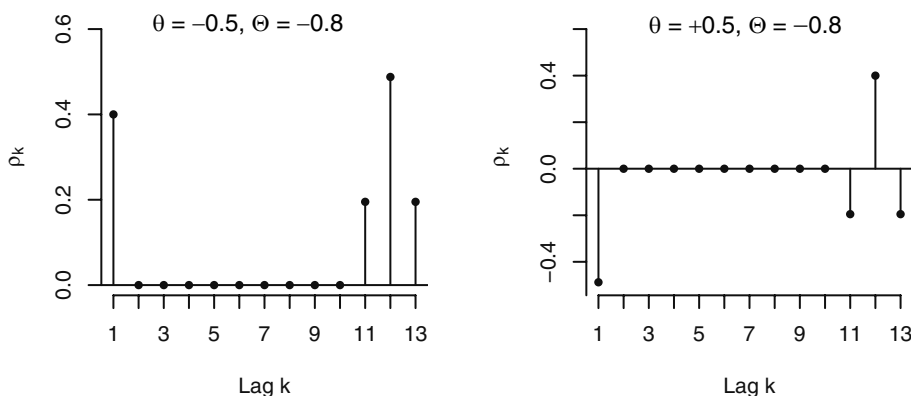
$$\rho_{11} = \rho_{13} = \frac{\theta \Theta}{(1 + \theta^2)(1 + \Theta^2)} \quad (10.2.4)$$

and

$$\rho_{12} = -\frac{\Theta}{1 + \Theta^2} \tag{10.2.5}$$

Exhibit 10.3 displays the autocorrelation functions for the model of Equation (10.2.1) with $\theta = \pm 0.5$ and $\Theta = -0.8$ as given by Equations (10.2.2)–(10.2.5).

Exhibit 10.3 Autocorrelations from Equations (10.2.2)–(10.2.5)



Of course, we could also introduce both short-term and seasonal autocorrelations by defining an MA model of order 12 with only θ_1 and θ_{12} nonzero. We shall see in the next section that the “multiplicative” model arises quite naturally for nonstationary models that entail differencing.

In general, then, we define a **multiplicative seasonal ARMA(p, q) \times (P, Q) $_s$ model with seasonal period s** as a model with AR characteristic polynomial $\phi(x)\Phi(x)$ and MA characteristic polynomial $\theta(x)\Theta(x)$, where

$$\left. \begin{aligned} \phi(x) &= 1 - \phi_1x - \phi_2x^2 - \dots - \phi_px^p \\ \Phi(x) &= 1 - \Phi_1x^s - \Phi_2x^{2s} - \dots - \Phi_Px^{Ps} \end{aligned} \right\} \tag{10.2.6}$$

and

$$\left. \begin{aligned} \theta(x) &= 1 - \theta_1x - \theta_2x^2 - \dots - \theta_qx^q \\ \Theta(x) &= 1 - \Theta_1x^s - \Theta_2x^{2s} - \dots - \Theta_Qx^{Qs} \end{aligned} \right\} \tag{10.2.7}$$

The model may also contain a constant term θ_0 . Note once more that we have just a special ARMA model with AR order $p + Ps$ and MA order $q + Qs$, but the coefficients are not completely general, being determined by only $p + P + q + Q$ coefficients. If $s = 12$, $p + P + q + Q$ will be considerably smaller than $p + Ps + q + Qs$ and will allow a much more parsimonious model.

As another example, suppose $P = q = 1$ and $p = Q = 0$ with $s = 12$. The model is then

$$Y_t = \Phi Y_{t-12} + e_t - \theta e_{t-1} \tag{10.2.8}$$

Using our standard techniques, we find that

$$\gamma_1 = \Phi\gamma_{11} - \theta\sigma_e^2 \quad (10.2.9)$$

and

$$\gamma_k = \Phi\gamma_{k-12} \text{ for } k \geq 2 \quad (10.2.10)$$

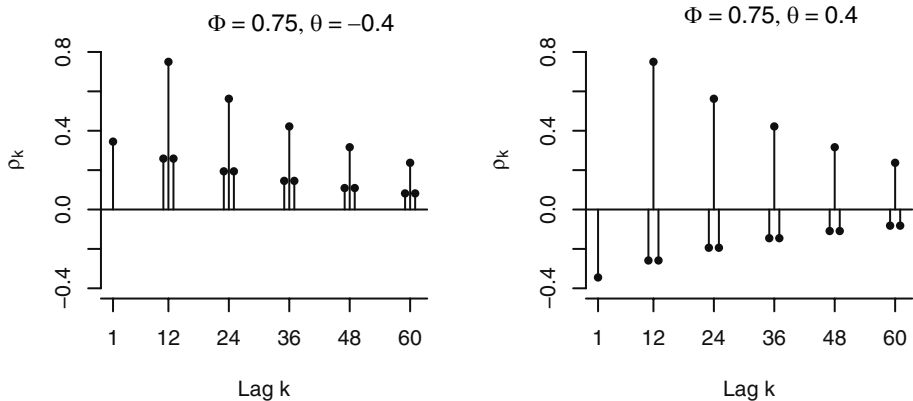
After considering the equations implied by various choices for k , we arrive at

$$\left. \begin{aligned} \gamma_0 &= \left[\frac{1+\theta^2}{1-\Phi^2} \right] \sigma_e^2 \\ \rho_{12k} &= \Phi^k \text{ for } k \geq 1 \\ \rho_{12k-1} &= \rho_{12k+1} = \left(-\frac{\theta}{1+\theta^2} \Phi^k \right) \text{ for } k = 0, 1, 2, \dots \end{aligned} \right\} \quad (10.2.11)$$

with autocorrelations for all other lags equal to zero.

Exhibit 10.4 displays the autocorrelation functions for two of these seasonal ARIMA processes with period 12: one with $\Phi = 0.75$ and $\theta = 0.4$, the other with $\Phi = 0.75$ and $\theta = -0.4$. The shape of these autocorrelations is somewhat typical of the sample autocorrelation functions for numerous seasonal time series. The even simpler autocorrelation function given by Equations (10.2.3), (10.2.4), and (10.2.5) and displayed in Exhibit 10.3 also seems to occur frequently in practice (perhaps after differencing).

Exhibit 10.4 Autocorrelation Functions from Equation (10.2.11)



10.3 Nonstationary Seasonal ARIMA Models

An important tool in modeling nonstationary seasonal processes is the seasonal difference. The **seasonal difference** of period s for the series $\{Y_t\}$ is denoted $\nabla_s Y_t$ and is defined as

$$\nabla_s Y_t = Y_t - Y_{t-s} \quad (10.3.1)$$

For example, for monthly series we consider the changes from January to January, February to February, and so forth for successive years. Note that for a series of length n , the seasonal difference series will be of length $n - s$; that is, s data values are lost due to seasonal differencing.

As an example where seasonal differencing is appropriate, consider a process generated according to

$$Y_t = S_t + e_t \quad (10.3.2)$$

with

$$S_t = S_{t-s} + \varepsilon_t \quad (10.3.3)$$

where $\{e_t\}$ and $\{\varepsilon_t\}$ are independent white noise series. Here $\{S_t\}$ is a “seasonal random walk,” and if $\sigma_\varepsilon \ll \sigma_e$, $\{S_t\}$ would model a slowly changing seasonal component.

Due to the nonstationarity of $\{S_t\}$, clearly $\{Y_t\}$ is nonstationary. However, if we seasonally difference $\{Y_t\}$, as given in Equation (10.3.1), we find

$$\begin{aligned} \nabla_s Y_t &= S_t - S_{t-s} + e_t - e_{t-s} \\ &= \varepsilon_t + e_t - e_{t-s} \end{aligned} \quad (10.3.4)$$

An easy calculation shows that $\nabla_s Y_t$ is stationary and has the autocorrelation function of an $\text{MA}(1)_s$ model.

The model described by Equations (10.3.2) and (10.3.3) could also be generalized to account for a nonseasonal, slowly changing stochastic trend. Consider

$$Y_t = M_t + S_t + e_t \quad (10.3.5)$$

with

$$S_t = S_{t-s} + \varepsilon_t \quad (10.3.6)$$

and

$$M_t = M_{t-1} + \xi_t \quad (10.3.7)$$

where $\{e_t\}$, $\{\varepsilon_t\}$, and $\{\xi_t\}$ are mutually independent white noise series. Here we take both a seasonal difference and an ordinary nonseasonal difference to obtain[†]

[†] It should be noted that $\nabla_s Y_t$ will in fact be stationary and $\nabla \nabla_s Y_t$ will be noninvertible. We use Equations (10.3.5), (10.3.6), and (10.3.7) merely to help motivate multiplicative seasonal ARIMA models.

$$\begin{aligned}\nabla\nabla_s Y_t &= \nabla(M_t - M_{t-s} + \varepsilon_t + e_t - e_{t-s}) \\ &= (\xi_t + \varepsilon_t + e_t) - (\varepsilon_{t-1} + e_{t-1}) - (\xi_{t-s} + e_{t-s}) + e_{t-s-1}\end{aligned}\quad (10.3.8)$$

The process defined here is stationary and has nonzero autocorrelation only at lags 1, $s-1$, s , and $s+1$, which agrees with the autocorrelation structure of the multiplicative seasonal model $\text{ARMA}(0,1)\times(0,1)$ with seasonal period s .

These examples lead to the definition of nonstationary seasonal models. A process $\{Y_t\}$ is said to be a **multiplicative seasonal ARIMA model** with nonseasonal (regular) orders p , d , and q , seasonal orders P , D , and Q , and seasonal period s if the differenced series

$$W_t = \nabla^d \nabla_s^D Y_t \quad (10.3.9)$$

satisfies an $\text{ARMA}(p,q)\times(P,Q)_s$ model with seasonal period s .[†] We say that $\{Y_t\}$ is an $\text{ARIMA}(p,d,q)\times(P,D,Q)_s$ model with seasonal period s .

Clearly, such models represent a broad, flexible class from which to select an appropriate model for a particular time series. It has been found empirically that many series can be adequately fit by these models, usually with a small number of parameters, say three or four.

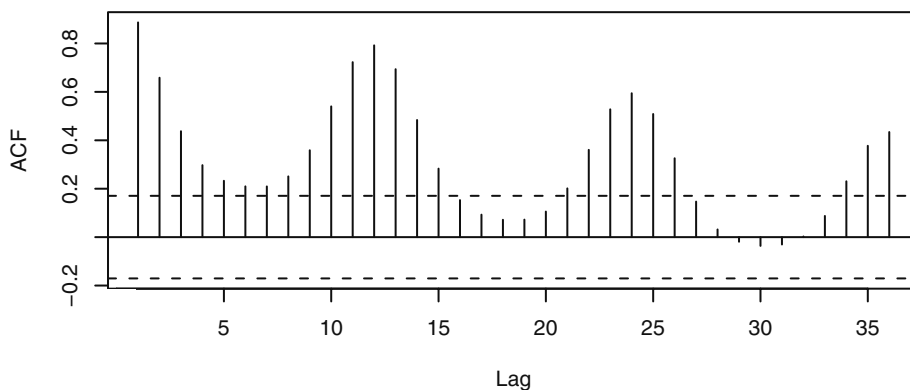
10.4 Model Specification, Fitting, and Checking

Model specification, fitting, and diagnostic checking for seasonal models follow the same general techniques developed in Chapters 6, 7, and 8. Here we shall simply highlight the application of these ideas specifically to seasonal models and pay special attention to the seasonal lags.

Model Specification

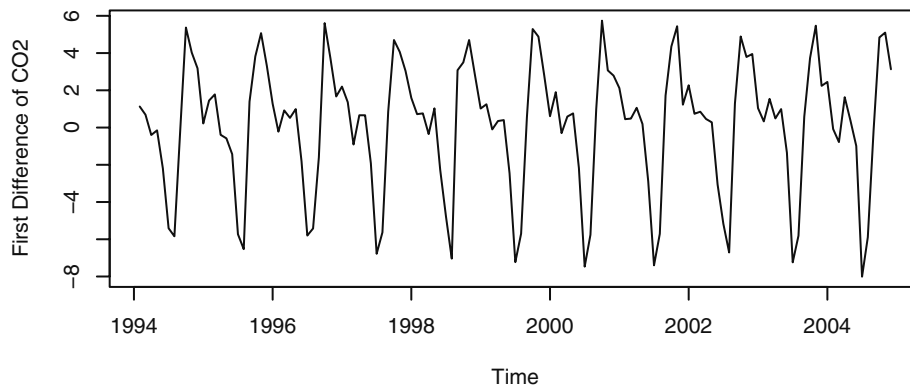
As always, a careful inspection of the time series plot is the first step. Exhibit 10.1 on page 227 displays monthly carbon dioxide levels in northern Canada. The upward trend alone would lead us to specify a nonstationary model. Exhibit 10.5 shows the sample autocorrelation function for that series. The seasonal autocorrelation relationships are shown quite prominently in this display. Notice the strong correlation at lags 12, 24, 36, and so on. In addition, there is substantial other correlation that needs to be modeled.

[†] Using the backshift operator notation of Appendix D, page 106, we may write the general $\text{ARIMA}(p,d,q)\times(P,D,Q)_s$ model as $\phi(B)\Phi(B)\nabla^d \nabla_s^D Y_t = \theta(B)\Theta(B)e_t$.

Exhibit 10.5 Sample ACF of CO₂ Levels

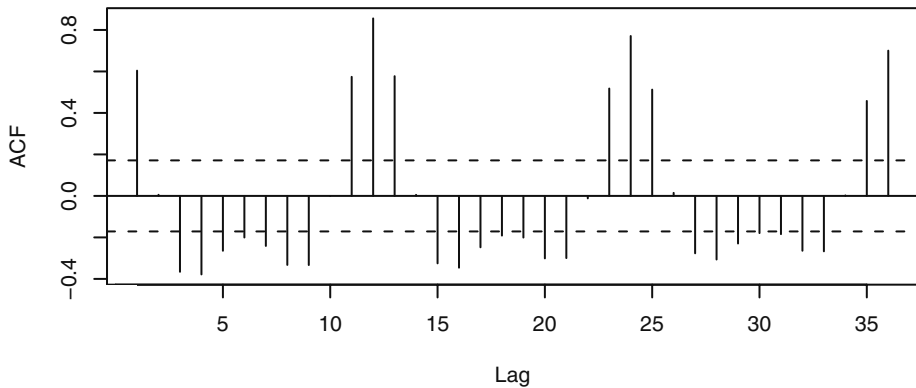
```
> acf(as.vector(co2), lag.max=36)
```

Exhibit 10.6 shows the time series plot of the CO₂ levels after we take a first difference.

Exhibit 10.6 Time Series Plot of the First Differences of CO₂ Levels

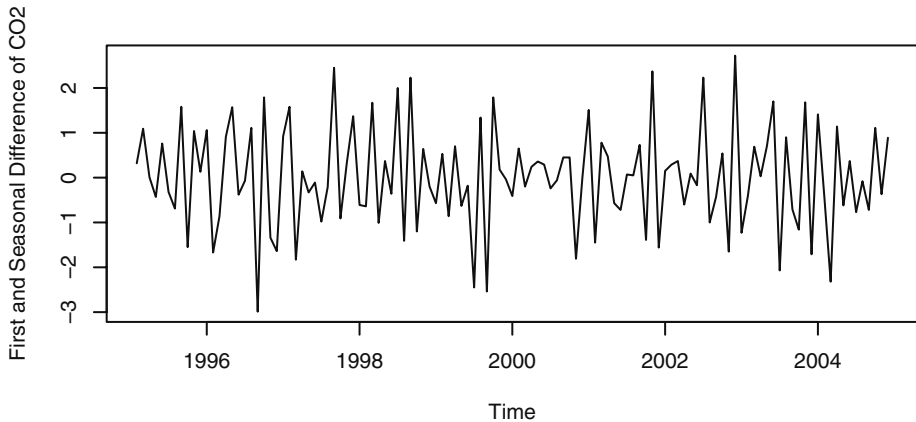
```
> plot(diff(co2), ylab='First Difference of CO2', xlab='Time')
```

The general upward trend has now disappeared but the strong seasonality is still present, as evidenced by the behavior shown in Exhibit 10.7. Perhaps seasonal differencing will bring us to a series that may be modeled parsimoniously.

Exhibit 10.7 Sample ACF of First Differences of CO₂ Levels


```
> acf(as.vector(diff(co2)), lag.max=36)
```

Exhibit 10.8 displays the time series plot of the CO₂ levels after taking both a first difference and a seasonal difference. It appears that most, if not all, of the seasonality is gone now.

Exhibit 10.8 Time Series Plot of First and Seasonal Differences of CO₂


```
> plot(diff(diff(co2), lag=12), xlab='Time',
       ylab='First and Seasonal Difference of CO2')
```

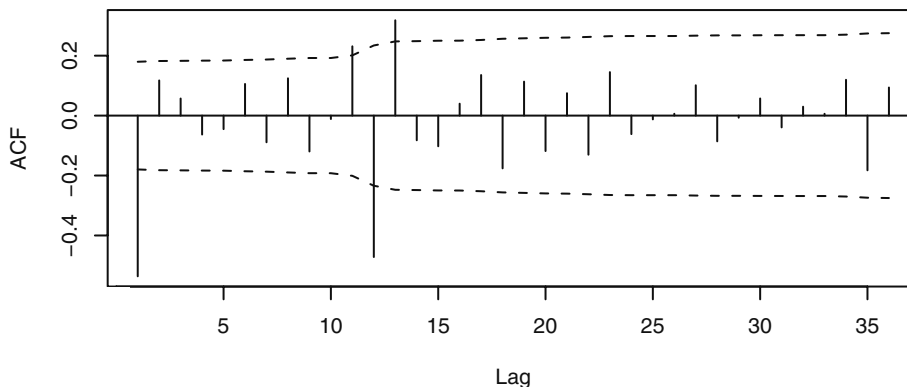
Exhibit 10.9 confirms that very little autocorrelation remains in the series after these two differences have been taken. This plot also suggests that a simple model which incorporates the lag 1 and lag 12 autocorrelations might be adequate.

We will consider specifying the multiplicative, seasonal ARIMA(0,1,1)×(0,1,1)₁₂ model

$$\nabla_{12}\nabla Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13} \quad (10.4.10)$$

which incorporates many of these requirements. As usual, all models are tentative and subject to revision at the diagnostics stage of model building.

Exhibit 10.9 Sample ACF of First and Seasonal Differences of CO₂



```
> acf(as.vector(diff(diff(co2), lag=12)), lag.max=36, ci.type='ma')
```

Model Fitting

Having specified a tentative seasonal model for a particular time series, we proceed to estimate the parameters of that model as efficiently as possible. As we have remarked earlier, multiplicative seasonal ARIMA models are just special cases of our general ARIMA models. As such, all of our work on parameter estimation in Chapter 7 carries over to the seasonal case.

Exhibit 10.10 gives the maximum likelihood estimates and their standard errors for the ARIMA(0,1,1)×(0,1,1)₁₂ model for CO₂ levels.

Exhibit 10.10 Parameter Estimates for the CO₂ Model

Coefficient	θ	Θ
Estimate	0.5792	0.8206
Standard error	0.0791	0.1137

$\hat{\sigma}_e^2 = 0.5446$; log-likelihood = -139.54, AIC = 283.08

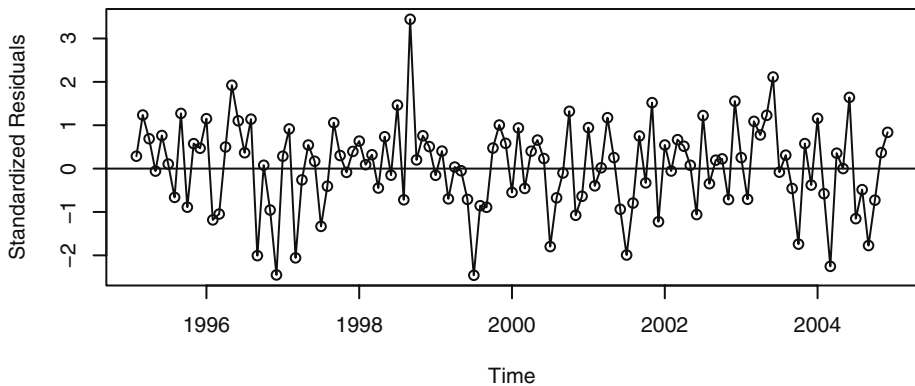
```
> m1.co2=arima(co2,order=c(0,1,1),seasonal=list(order=c(0,1,1),
  period=12))
> m1.co2
```

The coefficient estimates are all highly significant, and we proceed to check further on this model.

Diagnostic Checking

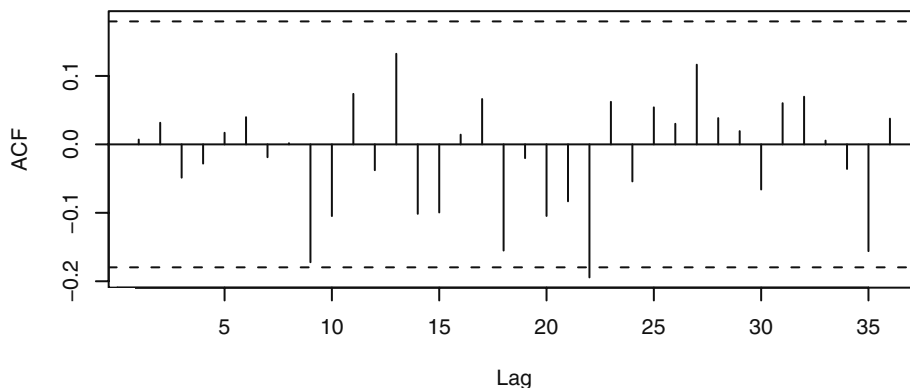
To check the estimated the $ARIMA(0,1,1) \times (0,1,1)_{12}$ model, we first look at the time series plot of the residuals. Exhibit 10.11 gives this plot for standardized residuals. Other than some strange behavior in the middle of the series, this plot does not suggest any major irregularities with the model, although we may need to investigate the model further for outliers, as the standardized residual at September 1998 looks suspicious. We investigate this further in Chapter 11.

Exhibit 10.11 Residuals from the $ARIMA(0,1,1) \times (0,1,1)_{12}$ Model



```
> plot(window(rstandard(m1.co2), start=c(1995,2)),
       ylab='Standardized Residuals', type='o')
> abline(h=0)
```

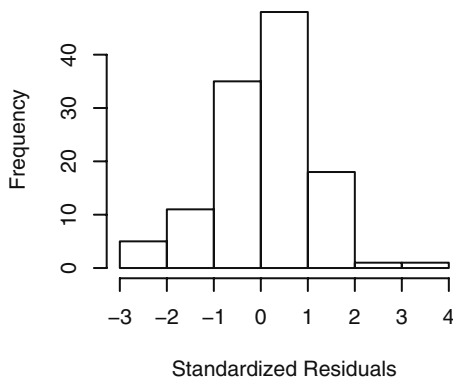
To look further, we graph the sample ACF of the residuals in Exhibit 10.12. The only “statistically significant” correlation is at lag 22, and this correlation has a value of only -0.17 , a very small correlation. Furthermore, we can think of no reasonable interpretation for dependence at lag 22. Finally, we should not be surprised that one autocorrelation out of the 36 displayed is statistically significant. This could easily happen by chance alone. Except for marginal significance at lag 22, the model seems to have captured the essence of the dependence in the series.

Exhibit 10.12 ACF of Residuals from the ARIMA(0,1,1)×(0,1,1)₁₂ Model

```
> acf(as.vector(window(rstandard(m1.co2), start=c(1995,2))),
      lag.max=36)
```

The Ljung-Box test for this model gives a chi-squared value of 25.59 with 22 degrees of freedom, leading to a p -value of 0.27—a further indication that the model has captured the dependence in the time series.

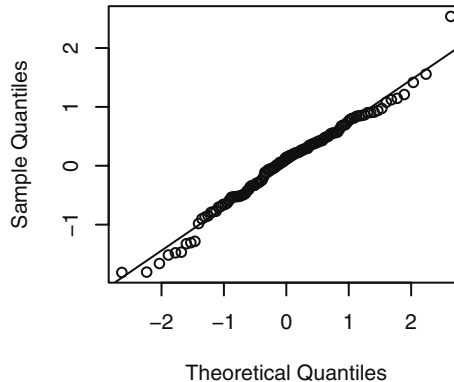
Next we investigate the question of normality of the error terms via the residuals. Exhibit 10.13 displays the histogram of the residuals. The shape is somewhat “bell-shaped” but certainly not ideal. Perhaps a quantile-quantile plot will tell us more.

Exhibit 10.13 Residuals from the ARIMA(0,1,1)×(0,1,1)₁₂ Model

```
> win.graph(width=3, height=3, pointsize=8)
> hist(window(rstandard(m1.co2), start=c(1995,2)),
      xlab='Standardized Residuals')
```

Exhibit 10.14 displays the QQ-normal plot for the residuals.

Exhibit 10.14 Residuals: ARIMA(0,1,1)×(0,1,1)₁₂ Model



```
> win.graph(width=2.5,height=2.5,points=8)
> qqnorm(window(rstandard(m1.co2),start=c(1995,2)))
> qqline(window(rstandard(m1.co2),start=c(1995,2)))
```

Here we again see the one outlier in the upper tail, but the Shapiro-Wilk test of normality has a test statistic of $W = 0.982$, leading to a p -value of 0.11, and normality is not rejected at any of the usual significance levels.

As one further check on the model, we consider overfitting with an ARIMA(0,1,2)×(0,1,1)₁₂ model with the results shown in Exhibit 10.15.

Exhibit 10.15 ARIMA(0,1,2)×(0,1,1)₁₂ Overfitted Model

Coefficient	θ_1	θ_2	Θ
Estimate	0.5714	0.0165	0.8274
Standard error	0.0897	0.0948	0.1224

$\hat{\sigma}_\epsilon^2 = 0.5427$: log-likelihood = -139.52, AIC = 285.05

```
> m2.co2=arima(co2,order=c(0,1,2),seasonal=list(order=c(0,1,1),
  period=12))
> m2.co2
```

When we compare these results with those reported in Exhibit 10.10 on page 237, we see that the estimates of θ_1 and Θ have changed very little—especially when the size of the standard errors is taken into consideration. In addition, the estimate of the new parameter, θ_2 , is not statistically different from zero. Note also that the estimate $\hat{\sigma}_\epsilon^2$ and the log-likelihood have not changed much while the AIC has actually increased.

The ARIMA(0,1,1)×(0,1,1)₁₂ model was popularized in the first edition of the seminal book of Box and Jenkins (1976) when it was found to characterize the logarithms of

a monthly airline passenger time series. This model has come to be known as the **airline model**. We ask you to analyze the original airline data in the exercises.

10.5 Forecasting Seasonal Models

Computing forecasts with seasonal ARIMA models is, as expected, most easily carried out recursively using the difference equation form for the model, as in Equations (9.3.28), (9.3.29) on page 199 and (9.3.40) on page 201. For example, consider the model $\text{ARIMA}(0,1,1) \times (1,0,1)_{12}$.

$$Y_t - Y_{t-1} = \Phi(Y_{t-12} - Y_{t-13}) + e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13} \quad (10.5.1)$$

which we rewrite as

$$Y_t = Y_{t-1} + \Phi Y_{t-12} - \Phi Y_{t-13} + e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13} \quad (10.5.2)$$

The one-step-ahead forecast from origin t is then

$$\hat{Y}_t(1) = Y_t + \Phi Y_{t-11} - \Phi Y_{t-12} - \theta e_t - \Theta e_{t-11} + \theta\Theta e_{t-12} \quad (10.5.3)$$

and the next one is

$$\hat{Y}_t(2) = \hat{Y}_t(1) + \Phi Y_{t-10} - \Phi Y_{t-11} - \Theta e_{t-10} + \theta\Theta e_{t-11} \quad (10.5.4)$$

and so forth. The noise terms $e_{t-13}, e_{t-12}, e_{t-11}, \dots, e_t$ (as residuals) will enter into the forecasts for lead times $\ell = 1, 2, \dots, 13$, but for $\ell > 13$ the autoregressive part of the model takes over and we have

$$\hat{Y}_t(\ell) = \hat{Y}_t(\ell-1) + \Phi \hat{Y}_t(\ell-12) - \Phi \hat{Y}_t(\ell-13) \text{ for } \ell > 13 \quad (10.5.5)$$

To understand the general nature of the forecasts, we consider several special cases.

Seasonal AR(1)₁₂

The seasonal AR(1)₁₂ model is

$$Y_t = \Phi Y_{t-12} + e_t \quad (10.5.6)$$

Clearly, we have

$$\hat{Y}_t(\ell) = \Phi \hat{Y}_t(\ell-12) \quad (10.5.7)$$

However, iterating back on ℓ , we can also write

$$\hat{Y}_t(\ell) = \Phi^{k+1} Y_{t+r-11} \quad (10.5.8)$$

where k and r are defined by $\ell = 12k + r + 1$ with $0 \leq r < 12$ and $k = 0, 1, 2, \dots$. In other words, k is the integer part of $(\ell-1)/12$ and $r/12$ is the fractional part of $(\ell-1)/12$. If our last observation is in December, then the next January value is forecast as Φ times the last observed January value, February is forecast as Φ times the last observed February

value, and so on. Two Januarys ahead is forecast as Φ^2 times the last observed January. Looking just at January values, the forecasts into the future will decay exponentially at a rate determined by the magnitude of Φ . All of the forecasts for each month will behave similarly but with different initial forecasts depending on the particular month under consideration.

Using Equation (9.3.38) on page 201 and the fact that the ψ -weights are nonzero only for multiple of 12, namely,

$$\psi_j = \begin{cases} \Phi^{j/12} & \text{for } j = 0, 12, 24, \dots \\ 0 & \text{otherwise} \end{cases} \quad (10.5.9)$$

we have that the forecast error variance can be written as

$$\text{Var}(e_t(\ell)) = \left[\frac{1 - \Phi^{2k+2}}{1 - \Phi^2} \right] \sigma_e^2 \quad (10.5.10)$$

where, as before, k is the integer part of $(\ell - 1)/12$.

Seasonal MA(1)₁₂

For the seasonal MA(1)₁₂ model, we have

$$Y_t = e_t - \Theta e_{t-12} + \theta_0 \quad (10.5.11)$$

In this case, we see that

$$\left. \begin{aligned} \hat{Y}_t(1) &= -\Theta e_{t-11} + \theta_0 \\ \hat{Y}_t(2) &= -\Theta e_{t-10} + \theta_0 \\ &\vdots \\ \hat{Y}_t(12) &= -\Theta e_t + \theta_0 \end{aligned} \right\} \quad (10.5.12)$$

and

$$\hat{Y}_t(\ell) = \theta_0 \text{ for } \ell > 12 \quad (10.5.13)$$

Here we obtain different forecasts for the months of the first year, but from then on all forecasts are given by the process mean.

For this model, $\psi_0 = 1$, $\psi_{12} = -\Theta$, and $\psi_j = 0$ otherwise. Thus, from Equation (9.3.38) on page 201,

$$\text{Var}(e_t(\ell)) = \begin{cases} \sigma_e^2 & 1 \leq \ell \leq 12 \\ (1 + \Theta^2)\sigma_e^2 & 12 < \ell \end{cases} \quad (10.5.14)$$

ARIMA(0,0,0)×(0,1,1)₁₂

The ARIMA(0,0,0)×(0,1,1)₁₂ model is

$$Y_t - Y_{t-12} = e_t - \Theta e_{t-12} \quad (10.5.15)$$

or

$$Y_{t+\ell} = Y_{t+\ell-12} + e_{t+\ell} - \Theta e_{t+\ell-12}$$

so that

$$\left. \begin{aligned} \hat{Y}_t(1) &= Y_{t-11} - \Theta e_{t-11} \\ \hat{Y}_t(2) &= Y_{t-10} - \Theta e_{t-10} \\ &\vdots \\ \hat{Y}_t(12) &= Y_t - \Theta e_t \end{aligned} \right\} \quad (10.5.16)$$

and then

$$\hat{Y}_t(\ell) = \hat{Y}_t(\ell-12) \text{ for } \ell > 12 \quad (10.5.17)$$

It follows that all Januarys will forecast identically, all Februarys identically, and so forth.

If we invert this model, we find that

$$Y_t = (1 - \Theta)(Y_{t-12} + \Theta Y_{t-24} + \Theta^2 Y_{t-36} + \cdots) + e_t$$

Consequently, we can write

$$\left. \begin{aligned} \hat{Y}_t(1) &= (1 - \Theta) \sum_{j=0}^{\infty} \Theta^j Y_{t-11-12j} \\ \hat{Y}_t(2) &= (1 - \Theta) \sum_{j=0}^{\infty} \Theta^j Y_{t-10-12j} \\ &\vdots \\ \hat{Y}_t(12) &= (1 - \Theta) \sum_{j=0}^{\infty} \Theta^j Y_{t-12j} \end{aligned} \right\} \quad (10.5.18)$$

From this representation, we see that the forecast for each January is an exponentially weighted moving average of all observed Januarys, and similarly for each of the other months.

In this case, we have $\psi_j = 1 - \Theta$ for $j = 12, 24, \dots$, and zero otherwise. The forecast error variance is then

$$\text{Var}(e_t(\ell)) = [1 + k(1 - \Theta)^2] \sigma_e^2 \quad (10.5.19)$$

where k is the integer part of $(\ell - 1)/12$.

ARIMA(0,1,1) × (0,1,1)₁₂

For the ARIMA(0,1,1) × (0,1,1)₁₂ model

$$Y_t = Y_{t-1} + Y_{t-12} - Y_{t-13} + e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta \Theta e_{t-13} \quad (10.5.20)$$

the forecasts satisfy

$$\left. \begin{aligned} \hat{Y}_t(1) &= Y_t + Y_{t-11} - Y_{t-12} - \theta e_t - \Theta e_{t-11} + \theta \Theta e_{t-12} \\ \hat{Y}_t(2) &= \hat{Y}_t(1) + Y_{t-10} - Y_{t-11} - \Theta e_{t-10} + \theta \Theta e_{t-11} \\ &\vdots \\ \hat{Y}_t(12) &= \hat{Y}_t(11) + Y_t - Y_{t-1} - \Theta e_t + \theta \Theta e_{t-1} \\ \hat{Y}_t(13) &= \hat{Y}_t(12) + \hat{Y}_t(1) - Y_t + \theta \Theta e_t \end{aligned} \right\} \quad (10.5.21)$$

and

$$\hat{Y}_t(\ell) = \hat{Y}_t(\ell-1) + \hat{Y}_t(\ell-12) - \hat{Y}_t(\ell-13) \quad \text{for } \ell > 13 \quad (10.5.22)$$

To understand the general pattern of these forecasts, we can use the representation

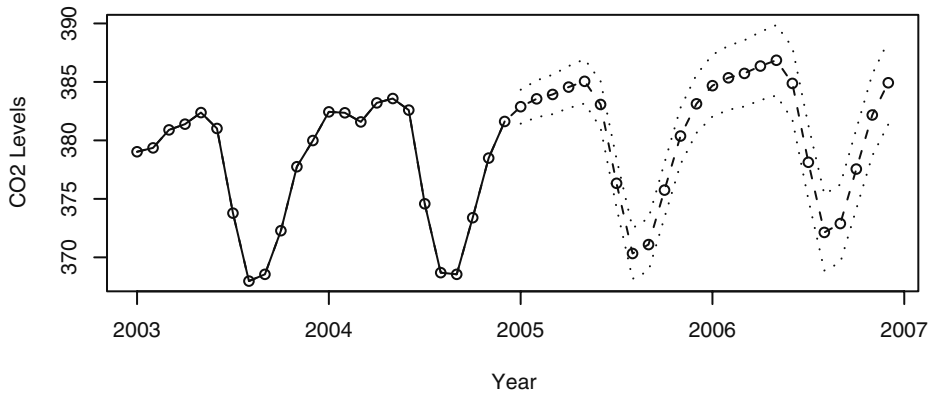
$$\hat{Y}_t(\ell) = A_1 + A_2 \ell + \sum_{j=0}^6 \left[B_{1j} \cos\left(\frac{2\pi j \ell}{12}\right) + B_{2j} \sin\left(\frac{2\pi j \ell}{12}\right) \right] \quad (10.5.23)$$

where the A 's and B 's are dependent on Y_t, Y_{t-1}, \dots , or, alternatively, determined from the initial forecasts $\hat{Y}_t(1), \hat{Y}_t(2), \dots, \hat{Y}_t(13)$. This result follows from the general theory of difference equations and involves the roots of $(1-x)(1-x^{12}) = 0$.

Notice that Equation (10.5.23) reveals that the forecasts are composed of a linear trend in the lead time plus a sum of periodic components. However, the coefficients A_i and B_{ij} are more dependent on recent data than on past data and will adapt to changes in the process as our forecast origin changes and the forecasts are updated. This is in stark contrast to forecasting with deterministic time trend plus seasonal components, where the coefficients depend rather equally on both recent and past data and remain the same for all future forecasts.

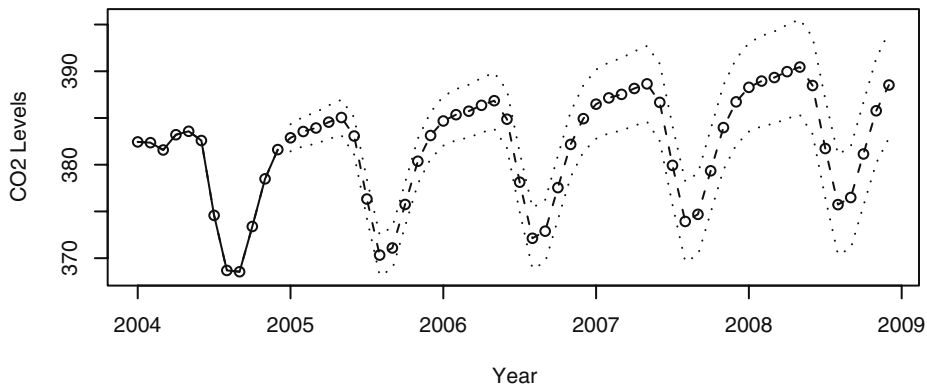
Prediction Limits

Prediction limits are obtained precisely as in the nonseasonal case. We illustrate this with the carbon dioxide time series. Exhibit 10.16 shows the forecasts and 95% forecast limits for a lead time of two years for the $\text{ARIMA}(0,1,1) \times (0,1,1)_{12}$ model that we fit. The last two years of observed data are also shown. The forecasts mimic the stochastic periodicity in the data quite well, and the forecast limits give a good feeling for the precision of the forecasts.

Exhibit 10.16 Forecasts and Forecast Limits for the CO₂ Model

```
> win.graph(width=4.875,height=3,pointsize=8)
> plot(m1.co2,n1=c(2003,1),n.ahead=24,xlab='Year',type='o',
      ylab='CO2 Levels')
```

Exhibit 10.17 displays the last year of observed data and forecasts out four years. At this lead time, it is easy to see that the forecast limits are getting wider, as there is more uncertainty in the forecasts.

Exhibit 10.17 Long-Term Forecasts for the CO₂ Model

```
> plot(m1.co2,n1=c(2004,1),n.ahead=48,xlab='Year',type='b',
      ylab='CO2 Levels')
```

10.6 Summary

Multiplicative seasonal ARIMA models provide an economical way to model time series whose seasonal tendencies are not as regular as we would have with a deterministic seasonal trend model which we covered in Chapter 3. Fortunately, these models are simply special ARIMA models so that no new theory is needed to investigate their properties. We illustrated the special nature of these models with a thorough modeling of an actual time series.

EXERCISES

10.1 Based on quarterly data, a seasonal model of the form

$$Y_t = Y_{t-4} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

has been fit to a certain time series.

(a) Find the first four ψ -weights for this model.

(b) Suppose that $\theta_1 = 0.5$, $\theta_2 = -0.25$, and $\sigma_e = 1$. Find forecasts for the next four quarters if data for the last four quarters are

Quarter	I	II	III	IV
Series	25	20	25	40
Residual	2	1	2	3

(c) Find 95% prediction intervals for the forecasts in part (b).

10.2 An AR model has AR characteristic polynomial

$$(1 - 1.6x + 0.7x^2)(1 - 0.8x^{12})$$

(a) Is the model stationary?

(b) Identify the model as a certain seasonal ARIMA model.

10.3 Suppose that $\{Y_t\}$ satisfies

$$Y_t = a + bt + S_t + X_t$$

where S_t is deterministic and periodic with period s and $\{X_t\}$ is a seasonal ARIMA($p, 0, q$) \times ($P, 1, Q$) $_s$ series. What is the model for $W_t = Y_t - Y_{t-s}$?

10.4 For the seasonal model $Y_t = \Phi Y_{t-4} + e_t - \theta e_{t-1}$ with $|\Phi| < 1$, find γ_0 and ρ_k .

10.5 Identify the following as certain multiplicative seasonal ARIMA models:

(a) $Y_t = 0.5Y_{t-1} + Y_{t-4} - 0.5Y_{t-5} + e_t - 0.3e_{t-1}$.

(b) $Y_t = Y_{t-1} + Y_{t-12} - Y_{t-13} + e_t - 0.5e_{t-1} - 0.5e_{t-12} + 0.25e_{t-13}$.

10.6 Verify Equations (10.2.11) on page 232.

- 10.7** Suppose that the process $\{Y_t\}$ develops according to $Y_t = Y_{t-4} + e_t$ with $Y_t = e_t$ for $t = 1, 2, 3$, and 4.
- (a) Find the variance function for $\{Y_t\}$.
 - (b) Find the autocorrelation function for $\{Y_t\}$.
 - (c) Identify the model for $\{Y_t\}$ as a certain seasonal ARIMA model.
- 10.8** Consider the Alert, Canada, monthly carbon dioxide time series shown in Exhibit 10.1 on page 227. The data are in the file named `co2`.
- (a) Fit a deterministic seasonal means plus linear time trend model to these data. Are any of the regression coefficients “statistically significant”?
 - (b) What is the multiple R-squared for this model?
 - (c) Now calculate the sample autocorrelation of the residuals from this model. Interpret the results.
- 10.9** The monthly airline passenger time series, first investigated in Box and Jenkins (1976), is considered a classic time series. The data are in the file named `airpass`.
- (a) Display the time series plots of both the original series and the logarithms of the series. Argue that taking logs is an appropriate transformation.
 - (b) Display and interpret the time series plots of the first difference of the logged series.
 - (c) Display and interpret the time series plot of the seasonal difference of the first difference of the logged series.
 - (d) Calculate and interpret the sample ACF of the seasonal difference of the first difference of the logged series.
 - (e) Fit the “airline model” (ARIMA(0,1,1)×(0,1,1)₁₂) to the logged series.
 - (f) Investigate diagnostics for this model, including autocorrelation and normality of the residuals.
 - (g) Produce forecasts for this series with a lead time of two years. Be sure to include forecast limits.
- 10.10** Exhibit 5.8 on page 99 displayed the monthly electricity generated in the United States. We argued there that taking logarithms was appropriate for modeling. Exhibit 5.10 on page 100 showed the time series plot of the first differences for this series. The filename is `electricity`.
- (a) Calculate the sample ACF of the first difference of the logged series. Is the seasonality visible in this display?
 - (b) Plot the time series of seasonal difference and first difference of the logged series. Does a stationary model seem appropriate now?
 - (c) Display the sample ACF of the series after a seasonal difference and a first difference have been taken of the logged series. What model(s) might you consider for the electricity series?

- 10.11** The quarterly earnings per share for 1960–1980 of the U.S. company Johnson & Johnson, are saved in the file named JJ.
- (a) Plot the time series and also the logarithm of the series. Argue that we should transform by logs to model this series.
 - (b) The series is clearly not stationary. Take first differences and plot that series. Does stationarity now seem reasonable?
 - (c) Calculate and graph the sample ACF of the first differences. Interpret the results.
 - (d) Display the plot of seasonal differences and the first differences. Interpret the plot. Recall that for quarterly data, a season is of length 4.
 - (e) Graph and interpret the sample ACF of seasonal differences with the first differences.
 - (f) Fit the model $\text{ARIMA}(0,1,1) \times (0,1,1)_4$, and assess the significance of the estimated coefficients.
 - (g) Perform all of the diagnostic tests on the residuals.
 - (h) Calculate and plot forecasts for the next two years of the series. Be sure to include forecast limits.
- 10.12** The file named boardings contains monthly data on the number of people who boarded transit vehicles (mostly light rail trains and city buses) in the Denver, Colorado, region for August 2000 through December 2005.
- (a) Produce the time series plot for these data. Be sure to use plotting symbols that will help you assess seasonality. Does a stationary model seem reasonable?
 - (b) Calculate and plot the sample ACF for this series. At which lags do you have significant autocorrelation?
 - (c) Fit an $\text{ARMA}(0,3) \times (1,0)_{12}$ model to these data. Assess the significance of the estimated coefficients.
 - (d) Overfit with an $\text{ARMA}(0,4) \times (1,0)_{12}$ model. Interpret the results.